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A new method for scalar instanton spectrum investigation

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Abstract. A new method is proposed for the investigation of the instanton structure of scalar field-theoretic models. It is based on rewriting the initial nonlinear field equation in spectral form and solving the latter by means of the spectral type linear equations.

1. Introduction

There has been great interest in the study of instantons in various Euclidean field theories. Recall that instantons are the non-trivial regular field equations with the action functional being stationary and finite. Great progress has been achieved in the study of instanton structure of gauge field theories. It was shown that the set of all instanton solutions could be divided into classes differing by values of topological quantum number $n = 0, 1, \ldots$ (Belavin *et al* 1975). Instantons of any number n can be interpreted as the solutions of a certain variational problem for the absolute minimum. This allows one to classify gauge instantons and develop a technique for their explicit construction.

There were several indications of the riches of an instanton structure in scalar theories (Lipatov 1976, Fubini 1976, Ushveridze 1977, 1979, 1980) but the absence of a general method made its investigation difficult. Several particular methods (e.g. the method of mechanical analogy (Lipatov 1977, Bukhvostov and Lipatov 1977, Ushveridze 1977, 1979)) work only for instantons whose equations can be reduced to one-dimensional form. We propose a general method that can be used, in principle, for the investigation of the instanton spectrum in scalar theories for any dimension.

We state our method for the rather wide class of Euclidean scalar field-theoretic models in the finite range, although it can be expanded for the infinite ranges too. It is based on the observation that all scalar instantons similar to gauge ones can be divided into classes differing by a certain whole number n. This 'quantum' number is not of the same nature as the topological one, but nevertheless for either of them the variational problem for the absolute minimum can be constructed, and instantons similar to the gauge theories are the solutions of these problems.

In § 2 we present our definitions and state the problem. In § 3 we formulate our method, which consists of two main parts. The first part is concerned with the transformation of the initial nonlinear field equation into spectral form, while in the second part we solve the spectral nonlinear equations by means of certain classes of spectral type linear equations, obtain the upper estimates etc. In § 4 the properties of the instantons are discussed. In § 5 we present the computational part of the method. In § 6 examples are shown to illustrate the method and the results of computer calculations are given.

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2. The problem

Consider the linear differential operator $\hat{L}$ of second order in the bounded closed range Ω (with the boundary Γ) of D-dimensional ($D \geqslant 1$) Euclidean space:

$$\hat{L} = -\sum_{\alpha,\beta=1}^{D} \frac{\partial}{\partial x_\alpha}\left(p_{\alpha\beta}\frac{\partial}{\partial x_\beta}\right) + q(x), \tag{1}$$

where $p_{\alpha\beta}(x)$ is a symmetric, positive definite in Ω, $D \times D$ matrix of continuously differentiable functions in Ω, $q(x)$ is a non-negative and continuous function in Ω.

Let A be the set of scalar functions (fields) $\phi(x)$ which are twice differentiable in Ω and satisfy one of the following two boundary conditions:

$$\phi|_\Gamma = 0 \qquad \text{or} \qquad \sum_{\alpha,\beta=1}^{D} p_{\alpha\beta} n_\beta \frac{\partial\phi}{\partial x_\alpha}\bigg|_\Gamma = 0, \tag{2}$$

where n_β is a normal unit vector on Γ. Hence, the operator $\hat{L}$ is Hermitian and positive on A.

The field-theoretic model of a scalar field ϕ is defined by its action functional:

$$S[\phi] = \tfrac{1}{2}\int_\Omega \phi\hat{L}\phi \, \mathrm{d}^D x - \frac{1}{\nu}\int_\Omega \omega(x)|\phi|^\nu \, \mathrm{d}^D x \equiv \tfrac{1}{2}S_2[\phi] - \frac{1}{\nu}S_\nu[\phi], \tag{3}$$

where $\omega(x)$ is a positive and continuous finite weight function in Ω, and ν is the degree of interaction nonlinearity ($\nu > 2$). (The function $\omega(x)$ includes the coupling constant.)

It is known (Nikolsky 1977) that if the degree of nonlinearity ν is related to the dimension D by the conditions

$$2 < \nu < 2D/(D-2) \qquad \text{for } D > 2 \tag{4a}$$

or

$$2 < \nu < \infty \qquad \text{for } 1 \leqslant D \leqslant 2 \tag{4b}$$

then for any function $\phi \in A$ the Sobolev inequality is satisfied:

$$(S_2[\phi])^{1/2} \geqslant C(S_\nu[\phi])^{1/\nu}, \tag{5}$$

where C is the positive number called the Sobolev constant and is independent of ϕ. (The value of C is completely defined by D, Ω, ν, $p_{\alpha\beta}(x)$, $q(x)$, $\omega(x)$ and constraint (2), and cannot be increased.) Throughout the paper condition (4) is supposed to be satisfied.

Let us now introduce the notion of instanton. By an instanton we mean any function $\phi \in A$ for which the action functional $S[\phi]$ is stationary (i.e. $\delta S[\phi] = 0$) and finite ($S[\phi] < \infty$), while an instanton spectrum is the set of all instantons ϕ and their actions $S[\phi]$.

The instantons in the model (3) are the solutions of the following nonlinear elliptic field equation:

$$\hat{L}\phi = \omega|\phi|^{\nu-2}\phi, \tag{6}$$

with the boundary constraints of type (2) and the condition of action finiteness which we write in the form

$$S = \frac{\nu-2}{2\nu}\int_\Omega \omega|\phi|^\nu \, \mathrm{d}^D x < \infty. \tag{7}$$

(Formula (7) can be obtained by multiplying both sides of (6) by ϕ, integrating and comparing with (3).)

Our aim is to investigate an instanton structure of the model (3), i.e. (1) to ascertain the set of instantons, (2) to classify this set and to predict some of their *a priori* properties and (3) to present the regular procedure for obtaining instantons and their actions numerically to any required accuracy.

3. The method

The method is divided into two parts. First, we transform the initial nonlinear equation (6) into the spectral nonlinear equation. For this purpose it is sufficient to use in (2), (6) and (7) a new function ψ,

$$\psi = \frac{\phi}{\{[2\nu/(\nu-2)]S\}^{1/\nu}}, \tag{8a}$$

and to introduce a new quantity

$$\mathscr{E} = \left(\frac{2\nu}{\nu-2}S\right)^{(\nu-2)/\nu} \tag{8b}$$

As a result we obtain the nonlinear spectral equation

$$\hat{L}\psi = \mathscr{E}\omega|\psi|^{\nu-2}\psi \tag{9}$$

with boundary constraints of type (2) and the additional integral condition

$$\int_\Omega \omega|\psi|^\nu \, \mathrm{d}^D x = 1. \tag{10}$$

It is easy to see that equation (9) with constraints (2) and (10) is the natural nonlinear generalisation of the linear spectral equations. (If $\nu = 2$ and $\omega(x) \equiv 1$, then (9) transforms to a Schrödinger type equation and (10) becomes the usual condition for wavefunction normalisation.) The solutions ψ and $\mathscr{E}$ of (9), in analogy with the linear case, we call eigenfunctions and eigenvalues, respectively, and the set of solutions we call the spectrum of (9).

The instanton spectrum of the initial model (3) can be easily reconstructed from the spectrum of equation (9) by means of formulae (8).

Let us now consider the second main part of our method: how to solve the spectral nonlinear equation (9). The idea is to construct the set of variational problems for the absolute minimum whose solutions are the eigenfunctions ψ and eigenvalues $\mathscr{E}$ of (9).

Consider the set B of functions $\rho(x)$ which are continuous in Ω and satisfy the single integral condition

$$\int_\Omega \omega|\rho|^\nu \, \mathrm{d}^D x = 1, \tag{11}$$

and for any $\rho(x)$ write the spectral linear equation

$$\hat{L}\psi = E\omega|\rho|^{\nu-2}\psi, \qquad \psi \in A, \tag{12}$$

with constraints (2) and (10).

Theorem 1. Let Ω be bounded and condition (4) be satisfied. Then the spectrum of (12) exists, and is positive and discrete. The proof can be found in Ushveridze (1981).

Let $E_n[|\rho(x)|]$ and $\psi_n[|\rho(x)|]$ be the nth eigenvalue and the corresponding nth eigenfunction of (12). (The usual ordering is assumed.) We consider $E_n[|\rho|]$ ($n = 0, 1, \ldots$) as a set of functionals defined on B.

Our central point, which is the basis of our proposed method, is formulated in the following theorem.

Theorem 2. If $\rho_{n,a}(x)$ is any function that makes the functional $E_n[|\rho|]$ stationary on B at given 'n', then

$$\psi_{n,a}(x) = \psi_n[|\rho_{n,a}(x)|] \qquad \text{and} \qquad \mathscr{E}_{n,a} = E_n[|\rho_{n,a}(x)|] \tag{13}$$

are the solutions of the nonlinear spectral equation (9). We give here the scheme of the proof.

Suppose the condition of the theorem is satisfied. Thus when substituting $\rho_{n,a}$ for $\rho'_{n,a} \in B$,

$$\rho'_{n,a} = \frac{\rho_{n,a} + \varepsilon f}{(\int_\Omega \omega |\rho_{n,a} + \varepsilon f|^\nu \, \mathrm{d}^D x)^{1/\nu}}, \tag{14}$$

the deviation in $E_n[|\rho|]$ must vanish to first order in ε for any $f(x)$. (It can be shown that for any n the eigenvalue $E_n[|\rho_{n,a}|]$ is non-degenerate.) When computing perturbatively (for the non-degenerate case), one can see that the deviation vanishes for any function f only if

$$|\rho_{n,a}(x)| = |\psi_n[|\rho_{n,a}(x)|]|. \tag{15}$$

Substitution of (15) in (12) shows that $\psi_{n,a} \equiv \psi_n[|\rho_{n,a}|]$ is the eigenfunction and $\mathscr{E}_{n,a} \equiv E_n[|\rho_{n,a}|]$ is the eigenvalue of the nonlinear spectral equation (9). Thus the statement is proved.

Let us now consider the relation (15), which can be interpreted as a functional equation for functions $|\rho_{n,a}|$. It is easy to see that the set of equations (15) supplemented by relations (13) is equivalent for $n = 0, 1, \ldots$ to the single equation (9). In fact, any solution of (15) leads to the solution of (9) and (*vice versa*) any solution of (9) corresponds to the solution of (15) for the particular n. This allows one to consider n as a 'quantum' number by means of which it is possible to divide the set G of all solutions of (9) into non-intersecting classes G_n (G_n is defined by the set of solutions of (15) for the particular n).

Theorem 3. Let Ω be bounded and condition (4) be satisfied. Then in any class G_n there exists at least one solution. This solution (we call it the main solution in the class G_n and use the notation $\psi_{n,0}$ and $\mathscr{E}_{n,0}$ for it) corresponds to the function $\rho_{n,0}$ that realises the absolute minimum of $E_n[|\rho|]$ on B. The proof can be found in Ushveridze (1981).

Hence we have shown that the boundedness of Ω and condition (4) are sufficient conditions for the existence of a countable instanton spectrum in the model (3).

Instantons and their actions that are related to $\psi_{n,a}$ and $\mathscr{E}_{n,a}$ by means of formulae (8) are denoted by $\phi_{n,a}$ and $S_{n,a}$, respectively, while we keep the notation G_n for the classes of $\phi_{n,a}$.

Until now, we did not know the inner structure of the G_n, and the nature of the number a that distinguishes solutions within G_n cannot be ascertained by means of our method.

4. Properties of instantons

As can be seen from formulae (8) and (13), all the properties of instantons and their actions originate from the corresponding properties of eigenfunctions and eigenvalues of linear spectral equations of type (12).

We show several typical properties.

(a) The node surfaces of $\psi_n \in G_n$ divide the range Ω into not more than $n+1$ parts for $D \geqslant 2$ and $n \geqslant 2$. When either $D=1$ and n is arbitrary or D is arbitrary and $n=0$ or $n=1$, these surfaces divide Ω into $n+1$ parts precisely (due to the oscillation theorem and its generalisation (Mikhlin 1968)).

(b) The sequence of the main eigenvalues $S_{n,0}$ does not decrease and $S_{0,0} = [(\nu-2)/2\nu]C^{2\nu/(\nu-2)}$.

The first statement follows from the chain

$$S_{n,0} \equiv \frac{\nu-2}{\nu}(\min_B E_n[|\rho|])^{\nu/(\nu-2)} \leqslant \frac{\nu-2}{2\nu}(\min_B E_{n+1}[|\rho|])^{\nu/(\nu-2)} \equiv S_{n+1,0}; \tag{16}$$

the second one follows from another chain

$$\begin{aligned} S_{0,0} &\equiv \frac{\nu-2}{2\nu}(\min_B E_0[|\rho|])^{\nu/(\nu-2)} \\ &= \frac{\nu-2}{2\nu}\left(\min_B \min_A \frac{\int_\Omega \phi\hat{L}\phi \, \mathrm{d}^D x}{\int_\Omega \omega|\rho|^{\nu-2}\phi^2 \, \mathrm{d}^D x}\right)^{\nu/(\nu-2)} \\ &= \frac{\nu-2}{2\nu}\left(\min_A \frac{(S_2[\phi])^{1/2}}{(S_\nu[\phi])^{1/\nu}}\right)^{\nu/(\nu-2)} = \frac{\nu-2}{2\nu}C^{2\nu/(\nu-2)}. \end{aligned} \tag{17}$$

(c) For all functions $\rho \in A$

$$S_{n,0} \leqslant \frac{\nu-2}{2\nu}(E_n[|\rho|])^{\nu/(\nu-2)}. \tag{18}$$

This inequality allows one to obtain the upper estimate on $S_{n,0}$ by solving a linear equation (12) for any convenient function $\rho(x)$.

(d) For large n

$$S_{n,a} \sim n^{2\nu/(\nu-2)D}. \tag{19}$$

To obtain this formula we use (8*b*) and an asymptotic expression for $E_n[|\rho|]$ in the large-n limit (see for example Courant and Hilbert (1931)).

5. Computational methods

As demonstrated in § 3, the problem of searching for the main instantons and their action is transformed to the problem of minimisation of functionals $E_n[|\rho|]$ on B. We use here the method of steepest descent.

For fixed n we consider the sequence $\{\rho^{(i)}(x)\}$ $(i=0,1,2,\ldots)$ defined by the recurrence relations

$$\rho^{(0)}\in B,$$
$$\rho^{(i+1)}=Q^{(i)}\left\{|\rho^{(i)}|^{\nu-2}\left(1-\varepsilon\int_\Omega \omega|\rho^{(i)}|^{\nu-2}\psi_n^2[|\rho^{(i)}|]\,\mathrm{d}^Dx\right)+\varepsilon|\psi_n[|\rho^{(i)}|]|^{\nu-2}\right\}^{1/(\nu-2)}\in B, \tag{20}$$

where $Q^{(i)}$ are normalising factors to satisfy the condition (11) for all $\rho^{(i)}$.

The sequence $\{\rho^{(i)}\}$ gives rise to the sequence of the nth eigenfunctions $\psi_n[|\rho^{(i)}|]$ and eigenvalues $E_n[|\rho^{(i)}|]$ of the linear equation (12).

It is easy to show, using perturbation theory, the Hölder inequality (Korn and Korn 1968) and conditions (10) and (11), that for sufficiently small ε the sequence of $E_n[|\rho^{(i)}|]$ decreases and that the transition from $\rho^{(i)}$ to $\rho^{(i+1)}$ defines the direction of steepest decrease in B of $E_n[|\rho|]$. More detailed analysis shows that the sequences

$$\phi^{(i)}=\psi_n[|\rho^{(i)}|](E_n[|\rho^{(i)}|])^{-1/(\nu-2)}, \qquad i=0,1,\ldots, \tag{21a}$$

and

$$S^{(i)}=\frac{\nu-2}{2\nu}(E_n[|\rho^{(i)}|])^{\nu/(\nu-2)}, \qquad i=0,1,\ldots, \tag{21b}$$

converge to the main instanton $\phi_{n,0}$ and its action $S_{n,0}$.

In order to find instantons for $n=0$ the following iterational method is more convenient:

$$\rho^{(0)}\in B, \qquad \rho^{(i+1)}=|\psi_0[|\rho^{(i)}|]|. \tag{22}$$

It is easy to show (using the abovementioned technique) that the corresponding sequence of $E_0[|\rho^{(i)}|]$ decreases. In fact,

$$\begin{aligned}
E_0[|\rho^{(i)}|]&=\frac{\int_\Omega \psi_0[|\rho^{(i)}|]\hat{L}\psi_0[|\rho^{(i)}|]\,\mathrm{d}^Dx}{\int_\Omega \omega|\rho^{(i)}|^{\nu-2}\psi_0^2[|\rho^{(i)}|]\,\mathrm{d}^Dx}\\
&>\frac{\int_\Omega \psi_0[|\rho^{(i)}|]\hat{L}\psi_0[|\rho^{(i)}|]\,\mathrm{d}^Dx}{\int_\Omega \omega|\rho^{(i+1)}|^{\nu-2}\psi_0^2[|\rho^{(i)}|]\,\mathrm{d}^Dx}\\
&>\min_A\frac{\int_\Omega \psi\hat{L}\psi\,\mathrm{d}^Dx}{\int_\Omega \omega|\rho^{(i+1)}|^{\nu-2}\psi^2\,\mathrm{d}^Dx}\\
&=\frac{\int_\Omega \psi_0[|\rho^{(i+1)}|]\hat{L}\psi_0[|\rho^{(i+1)}|]\,\mathrm{d}^Dx}{\int_\Omega \omega|\rho^{(i+1)}|^{\nu-2}\psi_0^2[|\rho^{(i+1)}|]\,\mathrm{d}^Dx}\\
&=E_0[|\rho^{(i+1)}|].
\end{aligned} \tag{23}$$

The convergence is guaranteed by its boundedness from below. At the same time the sequences (21a) and (21b) converge to $\phi_{0,0}$ and $S_{0,0}$ respectively.

6. Examples

We show here two examples. The first is one-dimensional and can be solved exactly:

$$S[\phi]=\frac{1}{2}\int_0^1\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2\mathrm{d}x-\frac{1}{4}\int_0^1\phi^4\,\mathrm{d}x,$$
$$\mathrm{d}^2\phi/\mathrm{d}x^2=-\phi^3, \qquad \phi(0)=\phi(1)=0. \tag{24}$$

The corresponding spectrum is the following:

$$\phi_{n,0}(x)=\frac{\sqrt{3}}{2(n+1)K(\frac{1}{2})}\mathrm{cn}([2(n+1)x-1]K(\tfrac{1}{2})|\tfrac{1}{2}),$$
$$S_{n,0}=\tfrac{4}{3}K^4(\tfrac{1}{2})(n+1)^4, \tag{25}$$

where $K(\frac{1}{2})$ is the whole elliptic integral (Abramowitz and Stegun 1964). We point out that there is only one solution in each class G_n.

Let us now investigate model (24) by means of our method and compare the results with (25). We seek the solution ϕ in the form of a finite series:

$$\phi_{n,0}(x)=\sum_{m=1}^{N}a_m^{(n,0)}\sin(m\pi x). \tag{26}$$

Hence, our method is reduced to an algebraic one, and all calculations can be performed on a computer. When increasing the number N and the number of steps N_{ST} in the iterational procedure (20) or (22), the solution of the problem can be found with any required accuracy.

The results of computation of the first few solutions are presented together with the exact values:

$N=3$, $N_{\mathrm{ST}}=30$, $S_{0,0}^{\mathrm{comp}}=15.7587$, $S_{0,0}^{\mathrm{exact}}=15.7561$,
$a_1^{(0,0)\mathrm{comp}}=4.4614\times10^{-1}$, $a_2^{(0,0)\mathrm{comp}}=1.3054\times10^{-6}$,
$a_3^{(0,0)\mathrm{comp}}=-1.9995\times10^{-2}$, $a_1^{(0,0)\mathrm{exact}}=4.4608\times10^{-1}$,
$a_2^{(0,0)\mathrm{exact}}=0$, $a_3^{(0,0)\mathrm{exact}}=-2.0107\times10^{-2}$,
$N=3$, $N_{\mathrm{ST}}=30$, $S_{1,0}^{\mathrm{comp}}=261.4786$, $S_{1,0}^{\mathrm{exact}}=252.0970$,
$a_1^{(1,0)\mathrm{comp}}=-1.6601\times10^{-2}$, $a_2^{(1,0)\mathrm{comp}}=2.2340\times10^{-1}$,
$a_3^{(1,0)\mathrm{comp}}=-9.4136\times10^{-4}$, $a_1^{(1,0)\mathrm{exact}}=0$,
$a_2^{(1,0)\mathrm{exact}}=2.2304\times10^{-1}$, $a_3^{(1,0)\mathrm{exact}}=0$.

In two dimensions we consider an example of the same type:

$$S[\phi]=\tfrac{1}{2}\int_0^3\mathrm{d}x_1\int_0^5\mathrm{d}x_2\left[\left(\frac{\partial\phi}{\partial x_1}\right)^2+\left(\frac{\partial\phi}{\partial x_2}\right)^2\right]-\tfrac{1}{4}\int_0^3\mathrm{d}x_1\int_0^5\mathrm{d}x_2\phi^4,$$
$$\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}=-\phi^3,\qquad \phi(0,x_2)=\phi(3,x_2)=\phi(x_1,0)=\phi(x_1,5)=0. \tag{27}$$

We seek the solution $\phi(x_1,x_2)$ in the form of a finite series:

$$\phi_{n,0}(x)=\sum_{l,m=1}^{N}b_{l,m}^{(n,0)}\sin\left(\frac{l\pi}{3}x_1\right)\sin\left(\frac{m\pi}{5}x_2\right). \tag{28}$$

We do not know whether exact solutions of this model exist, and present here only our results:

$N=3$, $N=50$, $S_{0,0}=0.04879$,
$b_{1,1}^{(0,0)}=2.4665\times10^{-2}$, $b_{1,2}^{(0,0)}=7.0004\times10^{-5}$, $b_{1,3}^{(0,0)}=-4.5098\times10^{-3}$,
$b_{2,1}^{(0,0)}=8.7070\times10^{-8}$, $b_{2,2}^{(0,0)}=7.0234\times10^{-10}$, $b_{2,3}^{(0,0)}=-3.5820\times10^{-8}$,
$b_{3,1}^{(0,0)}=-1.5813\times10^{-3}$, $b_{3,2}^{(0,0)}=7.2447\times10^{-6}$, $b_{3,3}^{(0,0)}=7.0819\times10^{-4}$,

$N=3, \qquad N=20, \qquad S_{1,0}=0.1817,$

$b_{1,1}^{(1,0)}=-4.225\times10^{-3}, \qquad b_{1,2}^{(1,0)}=3.532\times10^{-2}, \qquad b_{1,3}^{(1,0)}=6.076\times10^{-3},$

$b_{2,1}^{(1,0)}=2.146\times10^{-3}, \qquad b_{2,2}^{(1,0)}=3.004\times10^{-3}, \qquad b_{2,3}^{(1,0)}=1.644\times10^{-4},$

$b_{3,1}^{(1,0)}=6.026\times10^{-4}, \qquad b_{3,2}^{(1,0)}=-3.542\times10^{-3}, \qquad b_{3,3}^{(1,0)}=-7.695\times10^{-4}.$

All the computational work was done in the Institute of Physics of the Academy of Sciences of the Georgian SSR.

Finally we shall note that the proposed method can be expanded for the investigation of field-theoretic models in infinite ranges and for more complicated nonlinearities.

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